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On periodicity and low complexity of infinite permutations ☆

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Abstract

We define an infinite permutation as a sequence of reals taken up to value, or, equivalently, as a linear ordering of $\mathbb{N}$ or of $\mathbb{Z}$. We introduce and characterize periodic permutations; surprisingly, for each period t there is an infinite number of distinct t-periodic permutations. At last, we study a complexity notion for permutations analogous to subword complexity for words, and consider the problem of minimal complexity of non-periodic permutations. Its answer is not analogous to that for words and is different for the right infinite and the bi-infinite case.

1. Infinite permutations

Let S be a finite or countable ordered set: we shall consider S equal either to $\mathbb{N}$, or to $\mathbb{Z}$, or to $\{1, 2, \dots, n\}$ for some integer n. Let $\mathcal{A}_S$ be the set of all sequences of pairwise distinct reals defined on S. Let us define an equivalence relation $\sim$ on $\mathcal{A}_S$ as follows: let $a, b \in \mathcal{A}_S$, where $a = \{a_s\}_{s \in S}$ and $b = \{b_s\}_{s \in S}$; then $a \sim b$ if and only if for all $s, r \in S$ the inequalities $a_s < a_r$ and $b_s < b_r$ hold or do not hold simultaneously. An equivalence class from $\mathcal{A}_S / \sim$ is called an (S-)*permutation*. Thus, an S-permutation α can be interpreted as a sequence of reals taken up to value and defined by any of its representative sequences a; in this case, we write $\alpha = \overline{a}$. In particular, a $\{1, \dots, n\}$-permutation always has a representative with all values in $\{1, \dots, n\}$, i. e., can be identified with a usual permutation from S_n.

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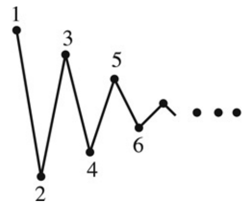

Fig. 1. A 2-periodic permutation.

In equivalent terms, a permutation can be defined as a linear ordering of S which may differ from the "natural" one. To distinguish the two orders, we shall write $i < j$, $i, j \in S$, for the natural order, and $\alpha_i < \alpha_j$ for the order induced by the permutation.

A permutation can be represented also by a diagram where the height of a point shows its position in respect of all other points.

Example 1. Let $a = \{a_i\}_{i=1}^{\infty} = 1, 0, 3/4, 1/4, 5/8, 3/8, \ldots$ be a sequence from $\mathcal{A}_{\mathbb{N}}$ defined by

$$a_i = \begin{cases} 1/2 + 1/2^k, & i = 2k - 1, \\ 1/2 - 1/2^k, & i = 2k. \end{cases}$$

Then the permutation $\overline{a} = \alpha_1, \ldots, \alpha_n, \ldots$ can be defined also by the linear ordering $\alpha_2 < \alpha_4 < \cdots < \alpha_{2i} < \alpha_{2i+2} \cdots < \alpha_{2i+3} < \alpha_{2i+1} < \cdots \alpha_1$ or by a diagram depicted at Fig. 1.

Infinite permutations have been considered e.g. in [3,4]. In fact, any sequence of reals occurring in any problem can be considered as a representative of some infinite permutation. We can also define an infinite permutation as a limit of a sequence of usual finite permutations. In this paper, we study the properties of infinite permutations which are defined similarly to properties of infinite words, such as periodicity and "subword" complexity. As we shall see below, some of these properties look similarly to those of infinite words and some are not.

A related problem of counting permutations arising from infinite words have been considered by Makarov [6]. At the same time, the general well-explored field of permutation patterns in the sense of [1] is not directly related to our problems.

2. Periodicity

Let us say that a permutation $\alpha = \{\alpha_s\}_{s \in S}$ is *t-periodic* if for all i and j such that $i, j, i + t, j + t \in S$ the inequalities $\alpha_i < \alpha_j$ and $\alpha_{i+t} < \alpha_{j+t}$ are equivalent. An $\mathbb{N}$-permutation is called *ultimately t-periodic* if there exists some N_0 such that these inequalities are equivalent provided that $i, j > N_0$. Note that the permutation from Example 1 is 2-periodic.

Recall that a bi-infinite word $w = \cdots w_{-1} w_0 w_1 \cdots$ on a finite alphabet Σ (with $w_i \in \Sigma$) is called *t-periodic* if $w_i = w_{i+t}$ for all i. Clearly, the number of t-periodic words is finite (and equal to $(\#\Sigma)^t$, if we fix the indices of symbols). So, it would be natural to conjecture that the number of t-periodic $\mathbb{Z}$-permutations is also not too large.

Surprisingly, for all $t \geq 2$ there exist infinitely many t-periodic $\mathbb{Z}$-permutations. A simple series of examples demonstrating it is given by a series of representatives $\{a(n)\}_{n=1}^{\infty}$, where the sequence $a(n)$ is

$$a(n) = \cdots - 1, 2n - 2, 1, 2n, 3, 2n + 2, \ldots.$$

Indeed, each of $a(n)$ is 2-periodic, and the first odd number which is greater than a given even one lies at the distance $2n-1$ from it. So, all permutations $\overline{a(n)}$ are different.

Let us characterize all t-periodic $\mathbb{Z}$-permutations and give a way to code each of them.

Let α be a t-periodic $\mathbb{Z}$-permutation, and a be any of its representative sequences of real numbers. For $i = 0, \ldots, t-1$, consider the permutation $A_i = \cdots \alpha_{i-t}, \alpha_i, \alpha_{i+t}, \alpha_{i+2t}, \cdots$. Each of them is monotonic (increasing or decreasing) in α. Let $S_i = (\inf_{k \equiv i \bmod t} a_k, \sup_{k \equiv i \bmod t} a_k)$ be the span of A_i, an open real interval. If two intervals S_i and S_j do not intersect, then the ordering between any element of A_i and any element of A_j is uniquely determined by the relative position of the intervals. Suppose that $S_i \cap S_j \neq \emptyset$. For definiteness, assume that the permutation A_i is increasing (the other case is treated similarly). Since $S_i = \bigcup_{n \in \mathbb{Z}} [a_{i+nt}, a_{i+(n+1)t}]$, for some m, n we have $a_{i+nt} < a_{j+mt} < a_{i+(n+1)t}$. Then, by periodicity, for every $k \in \mathbb{Z}$ we have $a_{i+nt+kt} < a_{j+mt+kt} < a_{i+(n+1)t+kt}$. Therefore, $S_i = S_j$, both sequences are increasing, and there is exactly one element of one of them between any two consecutive elements of the other.

Now we can code each t-periodic permutation in a unique way. First, let us partition the set $I = \{1, \ldots, t\}$ into one or more groups $I_1, \ldots, I_k$ (corresponding to disjoint intervals S_i taken in increasing order). Define $\alpha_x < \alpha_y$ whenever $x \equiv i (\bmod t)$, $y \equiv j (\bmod t)$, $i \in I_r$, $j \in I_s$, and $r < s$. It remains only to define the ordering on elements of progressions belonging to the same group; for every group this can be done separately.

Take any group $J = I_r$. Choose whether it will be *increasing* or *decreasing*; that is, whether every progression A_i from this group will be increasing or decreasing. Suppose that J is increasing (the other case is similar). Take the minimal remainder $i \in J$. For every $j \neq i$, $j \in J$, there must be exactly one integer n_j such that $\alpha_i < \alpha_{j+tn_j} < \alpha_{i+t}$. We can arbitrarily choose the values n_j, and then arbitrarily define the ordering of the numbers a_{j+tn_j} for all $j \neq i$, $j \in J$: all of them lie between α_i and α_{i+t} and the relations among them determine all relations among elements of A_{j_1}, A_{j_2} for all $j_1, j_2 \in J$. The permutation α is now defined.

We see that a t-periodic $\mathbb{Z}$-permutation can be uniquely represented by its code of the form

$$[i_{00}, i_{01}(n_{01}), \ldots, i_{0j_0}(n_{0j_0}), b_0] \cdots [i_{k0}, i_{k1}(n_{k1}), \ldots, i_{kj_k}(n_{kj_k}), b_k],$$

where all integers i_{rs} for $0 \leq r \leq k$, $0 \leq s \leq j_r$ are distinct and cover the set I; for each r, we have $i_{r0} < i_{rs}$ for $s > 0$; n_{rs} within the same range of r, s are arbitrary integers; at last, all $b_r \in \{<, >\}$. Here the brackets $[i_{r0}, i_{r1}(n_{r1}), \ldots, i_{rj_r}(n_{rj_r}), b_r]$ mean that the interval S_r corresponds to the group $I_r = \{i_{r0}, \ldots, i_{rj_r}\}$ which is increasing if $b_r = \{<\}$ and decreasing if $b_r = \{>\}$. The relations in that group are determined by the inequalities $\alpha_{i_{r0}} \; b_r \; \alpha_{i_{r1}+n_{r1}t} \; b_r \cdots b_r \; \alpha_{i_{rj_r}+n_{rj_r}t} \; b_r \; \alpha_{i_{r0}+t}$.

In particular, we see that the set of periodic permutations is countable.

Example 2. The code of the 5-periodic permutation α depicted at Fig. 2 is $[1, >]$ $[2, 5(2), 3(-1), <][4, <]$. One of the representatives on $\mathbb{R}$ of α is, e.g., the sequence $\{a_n\}_{n=1}^{\infty}$, where a_n is defined by

$$\begin{cases} a_{5k+1} = -k, \\ a_{5k+2} = 1 - \dfrac{1}{2^{3k+6}}, \\ a_{5k+3} = 1 - \dfrac{1}{2^{3k+11}}, \\ a_{5k+4} = k, \\ a_{5(k+1)} = 1 - \dfrac{1}{2^{3k+1}}. \end{cases}$$

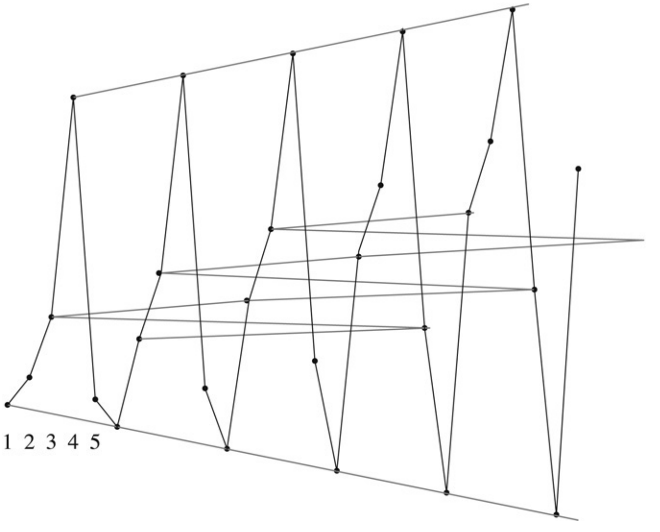

Fig. 2. A 5-periodic permutation.

In particular, this representative begins with

$$0, \frac{63}{64}, \frac{2047}{2048}, 1, \frac{1}{2}, -1, \frac{511}{512}, \ldots.$$

We see that here, a description of a permutation by a representative is not more illustrative than using a code or a graphical representation.

The above arguments show that in this way we obtain every t-periodic permutation of $\mathbb{Z}$. The number of these permutations is infinite since the integers n_{rs} can be arbitrary; at the same time, it is obviously countable.

It remains to mention that each t-periodic $\mathbb{N}$-permutation uniquely determines a t-periodic $\mathbb{Z}$-permutation: for $x, y \in \mathbb{Z}$, choose n such that $x + tn > 0$ and $y + tn > 0$, and set $\alpha_x < \alpha_y$ if and only if $\alpha_{x+tn} < \alpha_{y+tn}$. On the other hand, the non-negative positions of a t-periodic $\mathbb{Z}$-permutation form a t-periodic $\mathbb{N}$-permutation. So, it is a one-to-one correspondence and the characterization of periodic $\mathbb{N}$-permutations is induced by that for $\mathbb{Z}$-permutations.

At last, a natural question is: when does a periodic permutation admit a representation with integer (non-negative integer) values?

Lemma 1. *A t-periodic $\mathbb{N}$-permutation admits a representation with all values in $\mathbb{N}$ if and only if its code is of the form $[i_0, i_1(n_1), \ldots, i_t(n_t), <]$. A t-periodic $\mathbb{Z}$-permutation never admits such a representation.*

Proof. If there were at least two pairs of brackets in the code of a periodic $\mathbb{N}$-permutation, it would mean that all values from the first group lie between 0 and the least value from the second group. Since there is a finite number of integers in this interval, this is impossible for a permutation with all values in $\mathbb{N}$. For the same reason, the unique group cannot be decreasing: otherwise we would have to place an infinite number of integer values between one of the a_i, $0 < i \leq t$, and 0.

If we pass from $\mathbb{N}$-permutations to $\mathbb{Z}$-permutations, we similarly see that this unique group cannot be increasing: the infinite number of values with negative indices has no integers to be placed. Since any group must be either decreasing or increasing, this case is impossible. $\square$

Similarly, we can prove

Lemma 2. *A t-periodic $\mathbb{N}$-permutation admits a representation with all values in $\mathbb{Z}$ if and only if there are at most two pairs of brackets in its code, the first of them corresponding to a decreasing group and the second one to an increasing group. A t-periodic $\mathbb{Z}$-permutation admits a representation with all values in $\mathbb{Z}$ if and only if its code is of the form $[i_0, i_1(n_1), \ldots, i_t(n_t), a]$, where a can be equal either to $<$, or to $>$.* $\square$

3. Factors and complexity

Let us say that a finite permutation α' is a *factor* of an S-permutation $\alpha = \overline{a}$ if $\alpha' = \overline{a_k, a_{k+1}, \ldots, a_{k+n-1}}$ for some $n \geq 0$ and k such that $k, \ldots, k+n-1 \in S$. In what follows we denote the factor α' by $\alpha(n,k)$; note that $\alpha(n,k)$ is just a finite permutation of length n and can be treated as usual.

The notion of factor of a permutation is analogous to that of *factor*, or *subword*, of a word $w = \cdots w_1 w_2 \cdots$, $w_i \in \Sigma$, which is defined as a word $w_k w_{k+1} \cdots w_{k+n-1}$ for some allowable k and some $n \geq 0$. Let us start with a discussion on words.

The *subword complexity* $f_w(n)$ of a word w is the number of its distinct factors of length n. This function gives a classical non-algorithmic way to define complexity of a sequence of symbols and is well-explored (see survey [5]). In particular, clearly, it satisfies $1 \leq f_w(n) \leq (\#\Sigma)^n$ and is non-decreasing; the complexity of an ultimately periodic word is ultimately constant; we also have the following classical

Lemma 3. *If an infinite word w is not ultimately periodic, then $f_w(n)$ is strictly growing and satisfies $f_w(n) \geq n+1$.*

One-sided infinite words of minimal complexity are called *Sturmian* words [2] and have many non-trivial properties. The most famous of them is the *Fibonacci* word $abaababaabaababaababaabaab\cdots$ which can be constructed by iterating the morphism $a \mapsto ab, b \mapsto a$. Unlike a one-sided word, a bi-infinite word can have complexity $n+1$ even if it is quite simple, for example, equal to $\cdots 000010000 \cdots$.

Now let us return to permutations. Analogously to the subword complexity of words, let us define *complexity* $f_\alpha(n)$ of a permutation α as the number of its factors of length n:

$$f_\alpha(n) = \#\{\alpha(n,k) \mid k, k+1, \ldots, k+n-1 \in S\}.$$

Clearly, this function satisfies $1 \leq f_\alpha(n) \leq n!$ and is non-decreasing. In what follows we shall show that not all its properties are analogous to those of subword complexity for words: in particular, it is not necessarily strictly growing even if the permutation is not periodic, and the "minimal" possible complexity of non-periodic permutations is different for $\mathbb{N}$- and $\mathbb{Z}$-permutations. However, the first lemma is unified and analogous to the classical result for words.

Lemma 4. *Let α be a $\mathbb{Z}$ ($\mathbb{N}$-)permutation; then $f_\alpha(n) \leq C$ if and only if α is periodic (ultimately periodic).*

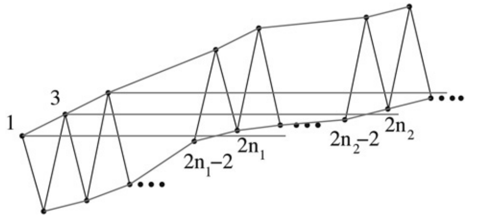

Fig. 3. A one-sided infinite permutation of low complexity.

Proof. The "if" part is obvious since ($\alpha_i < \alpha_j \iff \alpha_{i+t} < \alpha_{j+t}$) for all i, j (or, for $\mathbb{N}$-permutations, for $i, j \geq N_0$) implies $\alpha(n, i) = \alpha(n, i + t)$ for all n (for $\mathbb{N}$-permutations, this is again valid only if $i \geq N_0$). So, if α is (ultimately) periodic, then the number of different factors of length n in α is at most t if α is a $\mathbb{Z}$-permutation and at most $t + N_0 - 1$ if α is an $\mathbb{N}$-permutation. We see that this bound does not depend on n, which is what we need.

To prove the "only if" part, let us consider a permutation α whose complexity does not grow starting from the length n_0. Note that for all $k \geq 0$, the inequality $\alpha(n_0, i) \neq \alpha(n_0, j)$ implies $\alpha(n_0+k, i) \neq \alpha(n_0+k, j)$. Since $\#\{\alpha(n_0, i)\}_{i\in S} = \#\{\alpha(n_0+k, i)\}_{i\in S}$, the converse is also true: $\alpha(n_0, i) = \alpha(n_0, j)$ implies $\alpha(n_0 + k, i) = \alpha(n_0 + k, j)$. Since the set $\{\alpha(n_0, i)\}_{i\in S}$ is finite, there exist $i, j \in S$, $i < j$, such that $\alpha(n_0, i) = \alpha(n_0, j)$. We define $t = j - i$.

Suppose first that α is an $\mathbb{N}$-permutation. Let us prove that it is ultimately t-periodic starting from $N_0 = i$. Indeed, for all $l, m \geq i$ we can find some k such that α_l and α_m occur in $\alpha(n_0 + k, i)$ (i.e., $l, m < i + n_0 + k$). Since $\alpha(n_0, i) = \alpha(n_0, j) = \alpha(n_0, i + t)$, we have $\alpha(n_0 + k, i) = \alpha(n_0 + k, i + t)$. In particular, $\alpha_l < \alpha_m$ is equivalent to $\alpha_{l+t} < \alpha_{m+t}$. Since $l, m \geq i$ were chosen arbitrarily, this is what was to be proved.

The proof for $\mathbb{Z}$-permutations is analogous: the only additional remark is that in that case, we can uniquely continue equal words not only to the right but also to the left. □

The difference with the situation for words appears when we try to find minimal possible complexity of a non-periodic permutation.

Theorem 1. *For each unbounded increasing function $g(n)$ there exist an $\mathbb{N}$-permutation α which is not ultimately periodic and an integer N_0 such that $f_\alpha(n) \leq g(n)$ for all $n \geq N_0$.*

Proof. The needed permutation can be defined by the inequalities $\alpha_{2n-1} < \alpha_{2n+1}$ and $\alpha_{2n} < \alpha_{2n+2}$ for all $n \geq 1$, and $\alpha_{2n_k-2} < \alpha_{2k-1} < \alpha_{2n_k}$ for some sequence $\{n_k\}_{k=1}^{\infty}$ which grows sufficiently fast (see Fig. 3). Indeed, let us consider some sequence $\{n_k\}_{k=1}^{\infty}$ such that the difference $n_k - k$ is growing and compute $f_\alpha(n)$.

Note that since the difference $(n_k - k)$ is growing, any factor of α containing α_{2k-1} and α_{2n_k} occurs in α only once: $\alpha_{2k-1}, \alpha_{2n_k-2}$ and α_{2n_k} form the only triple of elements of α ordered as they are and standing at those distances from each other. So, the permutation α is not periodic.

Now note that if $n \leq 2n_{k+1} - 2(k + 1) + 1$, then there are only two distinct factors of the form $\alpha(n, i)$ with $i \geq 2k$ in α: one of them corresponds to odd is, and the other one to even is. In both of them, all elements which had an odd position in α are greater than all elements which had even positions.

Even if the first $2k-1$ factors of length n of α are distinct (which is the case for all sufficiently large n), we have $f_\alpha(n) \leq 2k + 1$ for all $n \leq 2n_{k+1} - 2(k + 1) + 1$.

Thus, to prove the theorem we just choose each of n_k as the least integer satisfying $g(2n_k - 2k + 2) > 2k + 1$ and $n_k - k > n_{k-1} - (k - 1)$ (the last condition is clearly equivalent to $n_k - n_{k-1} \geq 2$). $\square$

Theorem 2. *For each non-periodic $\mathbb{Z}$-permutation α we have $f_\alpha(n) \geq n - C$ for some constant C which can be arbitrarily large.*

Proof. Let us consider a non-periodic $\mathbb{Z}$-permutation α such that $f_\alpha(n) < n$ for some n. Since $f_\alpha(1) = 1$, there exists some $t \leq n$ such that $f_\alpha(t) \geq t$ but $f_\alpha(t + 1) < t + 1$. Since the function f_α is integer and non-decreasing, it follows that $f_\alpha(t) = f_\alpha(t + 1) = t$. This means that each of the t factors of α of length t can be uniquely extended to a factor of length $t + 1$ both to the left and to the right. Since the permutation α is bi-infinite, it follows that $\alpha(t + 1, i) = \alpha(t + 1, j)$ if and only if $i \equiv j (\text{mod } t)$. Thus, α is *t-quasi-periodic*, which by definition means that $(\alpha_i < \alpha_j) \Leftrightarrow (\alpha_{i+t} < \alpha_{j+t})$ for all i, j such that $| j - i | \leq t$. In particular this means that each of the t arithmetic progressions of difference t, i. e., permutations $A_i = \dots, \alpha_{i-t}, \alpha_i, \alpha_{i+t}, \dots, i = 1, \dots, t$, is monotonic (increasing or decreasing). Let us say that the permutations A_i and A_j, $i < j$, are *adjusted* if the permutation $A_{ij} = \dots, \alpha_{i-t}, \alpha_{j-t}, \alpha_i, \alpha_j, \alpha_{i+t}, \alpha_{j+t}, \dots$ is periodic.

Claim 1. *The permutation α is periodic if and only if all pairs of A_i and A_j for $0 \leq i < j \leq t$ are adjusted.*

Proof. If α is T-periodic, then it is also tT-periodic, and for each $n, m \in \mathbb{Z}$ the relation $\alpha_{i+nt} < \alpha_{j+mt}$ is equivalent to $\alpha_{i+nt+tT} < \alpha_{j+mt+tT}$. In A_{ij}, the distance between elements α_{i+nt} and $\alpha_{i+nt+tT}$, α_{j+mt} and $\alpha_{j+mt+tT}$ is equal to $2T$. So, A_{ij} is $2T$-periodic.

On the other hand, suppose that all A_{ij} are periodic, and let T be the lcm of their periods. Then all A_{ij} are T- and hence $2T$-periodic. Note that the element of A_{ij} which stands in it at the distance $2T$ from $\alpha_i + nt$ is $\alpha_{i+nt+tT}$, and the element of A_{ij} which stands in it at the distance $2T$ from α_{j+mt} is $\alpha_{j+mt+tT}$. This means that the relation $\alpha_{i+nt} < \alpha_{j+mt}$ for $i, j \in \{1, \dots, t\}$, $i \neq j$ is equivalent to $\alpha_{i+nt+tT} < \alpha_{j+mt+tT}$. This equivalence also holds for $i = j$ since A_i is monotonic. Thus, α is tT-periodic. $\square$

So, the fact that α is not periodic means that some $\beta = A_{ij}$ is not periodic. Here we fix i and j and set $\beta_1 = \alpha_i$; all other indices are uniquely determined by that.

Case 1. First, suppose that A_i is increasing and A_j is decreasing (or vice versa). If the intervals in which their representations lie do not intersect, then β is 2-periodic, a contradiction. If they intersect, then there exist some k and l such that either $\alpha_{j+(k+1)t} < \alpha_{i+lt} < \alpha_{j+kt}$, or $\alpha_{i+kt} < \alpha_{j+lt} < \alpha_{i+(k+1)t}$. These two cases can be considered similarly, and in both of them, the permutation α cannot be t-quasi-periodic. Indeed, if for instance $\alpha_{j+(k+1)t} < \alpha_{i+lt} < \alpha_{j+kt}$ and $l \leq k$, then $\alpha_{i+lt} < \alpha_{j+kt} \leq \alpha_{j+lt}$ but $\alpha_{i+(k+1)t} > \alpha_{i+lt} > \alpha_{j+(k+1)t}$, so that $\alpha_{i+lt} < \alpha_{j+lt}$ but $\alpha_{i+(k+1)t} > \alpha_{j+(k+1)t}$, contradicting to t-quasi-periodicity. Analogously if $l > k$, then $\alpha_{i+lt} > \alpha_{j+lt}$ but $\alpha_{i+kt} < \alpha_{j+kt}$. So, A_i and A_j cannot be monotonic in different directions.

Case 2. Now suppose that both A_i and A_j are increasing (the case when they are decreasing is analogous). Suppose that $\alpha_i < \alpha_j$ (and thus $\alpha_{i+kt} < \alpha_{j+kt}$ for all k due to quasi-periodicity). Let us consider a sequence of bi-infinite words $w(n)$, $n = 0, 1, \dots$, defined by

$$w(n)_k = \begin{cases} a, & \text{if } \alpha_{i+(k+n)t} < \alpha_{j+kt}, \\ b, & \text{otherwise.} \end{cases}$$

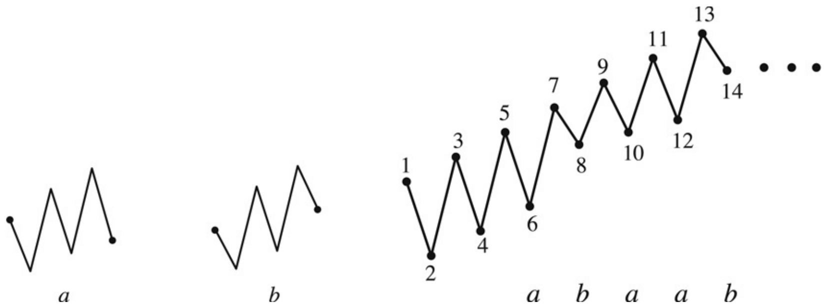

Fig. 4. A $\mathbb{Z}$-permutation of complexity at least $n-3$ based on the Fibonacci word.

By definition, $w(0) = \cdots aaaaa \cdots$. If all $w(n)$ are also equal to $\cdots aaaaa \cdots$, then the intervals of A_i and A_j do not intersect and β is 2-periodic, a contradiction. So, the letter b occurs in $w(m)$ for some m.

Case 2a. Suppose that some $w(m)$ is non-periodic. Then its subword complexity is at least $n+1$ due to Lemma 3. Let us count the number of factors of length N of α, where $N = (m+n+1)t + i - j$ for a fixed $n > 0$. This length is greater than $t+1$ and thus the factors starting with positions not equivalent modulo t are distinct since the t factors of α of length $t+1$ are arranged periodically.

Now let us consider some $\beta = \alpha(N, r)$ with $r \equiv R(\text{mod } t)$. Let k be the first integer such that $j + kt \geq r$. Then $\alpha(N, r)$ contains $\alpha_{j+kt}, \ldots, \alpha_{j+(k+n)t}$, and also $\alpha_{i+(k+m)t}, \ldots, \alpha_{i+(k+n+m)t}$. Thus, the elements of β whose positions in it are determined by R code a factor $w(m)_k w(m)_{k+1} \cdots w(m)_{k+n}$ of $w(m)$ of length $n+1$.

If we vary r but always choose it equal to R we shall mention all possible factors of $w(m)$ of length $n+1$ coded by them. These factors are at least $n+2$; so, there are at least $n+2$ factors of length N which start with positions equal to R modulo t in α. Summing them up for all the t possible values of R, we see that $f_\alpha(N) \geq t(n+2) = N - ((t-1)m + i - j)$.

Since successive integers which can be equal to N are situated at the distance t, we see that for all $M \geq 0$ there exists some $N \in \{M - t + 1, \ldots, M\}$ such that $N \equiv i - j \bmod t$. We have $f_\alpha(M) \geq f_\alpha(N) \geq N - ((t-1)m + i - j) \geq M - ((t-1)(m+1) + i - j)$. So, $f_\alpha(M) \geq M - c$ for all M; here c depends on m and t.

Example 3. Fig. 4 shows a permutation α of complexity $f_\alpha(n) \geq n-3$ built on the base of $w(3)$ equal to the (bi-infinite analogue of) the Fibonacci word. Note that we have some freedom in the relations between, e.g., α_5 and α_{12}, so it is easy to make the complexity greater than $n-3$.

Case 2b. It remains to consider the case when all $w(n)$ are periodic and some of them (say, $w(m)$) contain the letter b. Let $w(m)$ be p-periodic; so, letters b occur in it at most at the distance p from each other. Note also that $w(k)_s = b$ implies $w(k+1)_{s-1} = w(k+1)_s = b$, and thus $w(n) = \cdots bbbbb \cdots$ for all $n \geq m + p - 1$. So, there is just a finite number of sequences $w(n)$ not equal to $\cdots bbbbb \cdots$; all of them are periodic; let t_n be the period of $w(n)$.

Let us check that β is periodic with period $T = 2\,\mathrm{lcm}_{n\in\mathbb{N}} t_n$; as we have shown, T exists and is equal to $2\,\mathrm{lcm}(t_1, \ldots, t_{m+p-2})$. Indeed, let us consider β_s and β_r, where $s \equiv r \bmod T$. Since T is even, $\beta_s = \alpha_{i+kt}$ and $\beta_r = \alpha_{i+lt}$ or $\beta_s = \alpha_{j+kt}$ and $\beta_r = \alpha_{j+lt}$. Let us prove that for all p the inequality $\beta_s < \beta_{s+p}$ (*) holds if and only if $\beta_r < \beta_{r+p}$ (**); due to the symmetry, it is sufficient to consider $p > 0$.

If $\beta_s = \alpha_{i+kt}$ and $\beta_r = \alpha_{i+lt}$ then for all $p > 0$ both inequalities (*) and (**) clearly hold. If $\beta_s = \alpha_{j+kt}, \beta_r = \alpha_{j+lt}$ and p is even, $p = 2p'$, then $\beta_{s+p} = \alpha_{j+(k+p')t}, \beta_{r+p} = \alpha_{j+(l+p')t}$, and the inequalities also hold since the permutation A_j is increasing. At last, if p is odd, $p = 2p' - 1$, then $\beta_{s+p} = \alpha_{i+kt+p't}$ and $\beta_{r+p} = \alpha_{i+lt+p't}$. So, (*) holds if and only if $w(p')_k = b$ and (**) holds if and only if $w(p')_l = b$; but since $k \equiv l \bmod t_{p'}$, these conditions are equivalent. So, in all cases (*) is equivalent to (**), and the permutation β is periodic.

Example 4. The permutation α defined for all $i \in \mathbb{Z}$ by the inequalities $\alpha_{i+2} > \alpha_i$, $\alpha_{4i+1} < \alpha_{4i+3} < \alpha_{4i+2} < \alpha_{4i+4}$, $\alpha_{4i+2} > \alpha_{4i+5}$ and $\alpha_{4i+4} < \alpha_{4i+7}$ is 2-quasi-periodic, not 2-periodic but 4-periodic. Its code is $(1, 2(-1), 4(-1), 3(0), <)$. If we choose $i = 1$ and $j = 0$, then $w(0) = \cdots aaaaaa \cdots$, $w(1) = \cdots bababa \cdots$ and $w(2) = \cdots bbbbbb \cdots$.

We have considered all possible cases and shown that a permutation α with $f_\alpha(n) < n$ for some n can be non-periodic only in the case 2a. In that case, $f_\alpha(n) \geq n - c$ for some c and for all n. The theorem is proved. $\square$

Remark 1. The idea of Example 3 can be easily generalized to a family of examples of permutations of complexity $n - C$ with arbitrarily large C. To construct it, we just increase the length of factors which code symbols a and b and fix the relation between odd and even elements to be the same in all vague cases. More precisely, we fix a sequence w on $\{a, b\}$ of complexity $n + 1$, fix a positive integer $K > 1$ and define α by the relations

$\alpha_k < \alpha_l$	for all k and l such that $k < l$ and $l - k$ is odd;
$\alpha_{2k+1} > \alpha_{2k+2l}$	for all $l < K$;
$\alpha_{2k+1} > \alpha_{2k+2K}$	if and only if $w_k = a$;
$\alpha_{2k+1} > \alpha_{2k+2K+2l}$	for all $l > 0$ such that $w_k = w_{k+1} = \cdots w_{k+l} = a$.

It is not difficult to check that the relations above completely define the permutation α, and that a factor of this permutation is completely determined by the parity of its beginning and the underlying factor of w. The proof that $f_\alpha(n) = n - 2K + 3$ is left to the reader.

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